International Journal of Emerging Trends in Engineering and Basic Sciences (IJEEBS)
ISSN (Online) 2349-6967
Volume 3, Issue 3 (May-June 2016), PP.007-012

Thermoelastic Approach to Solid Circular Elliptical Cylinder

V. R. Manthena
(Department of Mathematics, Priyadarshini J. L. College of Engineering, Nagpur/RTMNU, India)

Abstract :- This paper is concerned with the determination of temperature change, displacement and thermal stresses of an elliptical finite solid circular cylinder with the help of Mathieu function and integral transform technique. It is assumed that length 2L subjected to a general prescribed heat flux on its boundary at $r = b$in the form of $Qf(\phi)g(z)H(t)$ and whose ends at $z = \pm L$ are exposed to free convection (h, T_∞) .The governing heat conduction equation has been solved by using Integral transform technique. The results are obtained in series form in terms of Bessel's functions. As a special case, aluminum metallic cylinder has been considered and the results for temperature change, displacement and stresses have been computed numerically.
Keywords: - Solid Circular Elliptical Cylinder, thermoelastic problem, Mathieu function, Integral transform.

I. INTRODUCTION

Choubey, N.K. , Geeta Shrivastava [1] and Mehta [3] have studied the problem of heat conduction and investigated the unknown temperature of elliptical cylinder using finite Fourier sine transform technique.

In this paper, an attempt has been made to determine the temperature distribution and unknown temperature of elliptical cylinder with known radiation type boundary conditions, using Mathiue function and Marchi-Fasulo transform technique.

II. STATEMENT OF THE PROBLEM

The heat conduction equation for elliptical cylinder is

$$\left(\frac{2d^2}{\cos(2\xi)-\cos(2\eta)}\right)\left(\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}\right)+\frac{\partial^2\theta}{\partial z^2}=\frac{\partial\theta}{\partial t} \tag{1}$$

Subject to the interior condition

$$\theta(\xi,\eta,z) = f(\eta,z) \qquad 0\le\xi\le\xi_0\,,\ -h\le z\le h \quad \text{(known)} \tag{2}$$

and initial condition

$$\theta = 0 \text{ at } t = 0 \tag{3}$$

The boundary conditions are

$$\left[h\theta(\xi,\eta,z)+k\frac{\partial\theta(\xi,\eta,z)}{\partial r}\right]_{\xi=b} = Qf(\eta)g(z)H(t) \tag{4}$$

$$\left[h\theta(\xi,\eta,z)+k\frac{\partial\theta(\xi,\eta,z)}{\partial z}\right]_{z=\pm L}=0 \quad (5)$$

Here we assume plane strain condition $\epsilon_{zz}=\epsilon_{rz}=\epsilon_{\phi z}=0$

The Stress- strain relations are given by

$$\sigma_{rr}=\frac{1}{E}\left[\sigma_{rr}-v(\sigma_{\phi\phi}+\sigma_{zz})\right]+\alpha\theta \quad (6)$$

$$\sigma_{\phi\phi}=\frac{1}{E}\left[\sigma_{\phi\phi}-v(\sigma_{rr}+\sigma_{zz})\right]+\alpha\theta \quad (7)$$

$$\sigma_{zz}=v\left(\sigma_{rr}+\sigma_{\phi\phi}\right)-E\alpha\theta \quad (8)$$

Solving for stresses in terms of strains, yields

$$\left.\begin{aligned}\sigma_{rr}&=\frac{E}{(1+v)(1-2v)}\left[(1-v)\epsilon_{rr}+v\epsilon_{\phi\phi}-(1+v)\alpha\theta\right] \quad (9)\\ \sigma_{\phi\phi}&=\frac{E}{(1+v)(1-2v)}\left[(1-v)\epsilon_{\phi\phi}+v\epsilon_{rr}-(1+v)\alpha\theta\right]\end{aligned}\right\}$$

And the strain –displacement relations, with u being the radial displacement, are

$$\epsilon_{rr}=\frac{du}{dr}\quad,\quad\epsilon_{\phi\phi}=\frac{u}{r} \quad (10)$$

Substituting Eq. (10) into Eq. (9), we obtain

$$\left.\begin{aligned}\sigma_{rr}&=\frac{E}{(1+v)(1-2v)}\left[(1-v)\frac{du}{dr}+v\frac{u}{r}-(1+v)\alpha\theta\right]\\ \sigma_{\phi\phi}&=\frac{E}{(1+v)(1-2v)}\left[(1-v)\frac{u}{r}+v\frac{du}{dr}-(1+v)\alpha\theta\right] \quad (11)\end{aligned}\right\}$$

Here k is the radiation constant on the plane surface of the cylinder. Equations (1) and (11) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Assuming $g(\xi)$ to be symmetric with respect to $\xi=0$, Taking Laplace transforms of Eq. (1)

$$\Delta_1\bar{\theta}^*=s\bar{\theta}^* \quad (12)$$

$$\frac{\partial\bar{\theta}^*}{\partial\rho}+\bar{h}\bar{\theta}^*=\frac{1}{s}f(\phi)g(\xi) \quad \text{at} \quad \rho=1$$

$$\frac{\partial\bar{\theta}^*}{\partial\xi}\pm\bar{h}\bar{\theta}^*=0 \quad \text{at} \quad \xi=\pm\xi_1 \quad (13)$$

Where Laplace transforms are indicated by an asterisk and s is the Laplace transforms parameter, and

$$\Delta_1 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\xi^2}$$

A general solution of Eq. (12) Subjected to boundary conditions (13) is

$$\bar{\theta}^* = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{(A_n\cos n\phi + B_n\sin n\phi)J_n(\mu\rho)\,C_k\cos\beta_k\xi}{s[(n+\bar{h})J_n(\mu) - \mu J_{n+1}(\mu)]} \quad (14)$$

Where J_n is Bessel function of the first kind and $n-th$ order ,

$\mu^2 = -(s+\beta^2)$, and $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)d\phi$

$$(A_n, B_n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\,(\cos n\phi, \sin n\phi)d\phi$$

$$C_n = \frac{\beta_k^2 + \bar{h}^2}{\bar{h} + \xi_1(\beta_k^2 + \bar{h}^2)}\int_{-\xi_1}^{\xi_1} g(\xi)\,\cos\beta_k\xi d\xi$$

And β_k are the roots of the equation

$$\beta\sin\beta\xi_1 - \bar{h}\cos\beta\xi_1 = 0 \quad (15)$$

The inverse Laplace transforms of Eq. (14) gives the temperature distribution in the cylinder

$$\bar{\theta} = \sum_{k=1}^{\infty} C_k\cos\beta_k\xi\sum_{n=0}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k+) + \beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm}\rho)\frac{\exp[-(\mu_{nm}^2 + \beta_k^2)\tau]}{(\mu_{nm}^2+\beta_k^2)\{[n(n+\bar{h})/\mu_{nm} - \mu_{nm}]J_n(\mu_{nm}) - \bar{h}J_{n+1}(\mu_{nm})\}}\right]$$

$$\bar{\theta} = \sum_{k=1}^{\infty} C_k\cos\beta_k\xi\sum_{n=0}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k+) + \beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm}\rho)\frac{\exp[-(\mu_{nm}^2 + \beta_k^2)\tau]}{(\mu_{nm}^2+\beta_k^2)\{[n(n+\bar{h})/\mu_{nm} - \mu_{nm}]J_n(\mu_{nm}) - \bar{h}J_{n+1}(\mu_{nm})\}}\right]$$

(16)

Where I_n the modified Bessel is function of the first kind and $n-th$ order, and μ_{nm} is the m-th positive root of the equation

$$(n+h)J_n(\mu_{nm}) - \mu_{nm}J_{n+1}(\mu_{nm}) = 0$$

Using Eq. (5) in equilibrium equation (3) and simplifying, results in the equilibrium equation in terms of displacement u as

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ur)}{dr}\right] = \frac{1+v}{1-v}\alpha\frac{d\theta}{dr} \quad (17)$$

The integration of Eq. (17) yields

$$u = \frac{1+v}{1-v}\frac{\alpha}{r}\int_0^r \bar{\theta} r dr + C_1 r + \frac{C_2}{r} \quad (18)$$

Where C_1 and C_2 are the constants of integration. Since the displacement must be finite at $r = 0$, it follows that C_2 must be zero. The components of strain are obtained as

$$\epsilon_{rr} = -\frac{1+v}{1-v}\frac{\alpha}{r^2}\int_0^r \bar{\theta} r dr + C_1 + \frac{1+v}{1-v}\alpha\bar{\theta}$$

$$\epsilon_{\phi\phi} = \frac{1+v}{1-v}\frac{\alpha}{r^2}\int_0^r \bar{\theta} r dr + C_1 \quad (19)$$

And the stresses from Eq. (11) are

$$\sigma_{rr} = -\frac{E}{(1-v)}\frac{\alpha}{r^2}\int_0^r \bar{\theta} r dr + C_1 \frac{E}{(1+v)(1-2v)}$$

$$\sigma_{\phi\phi} = \frac{E}{(1-v)}\frac{\alpha}{r^2}\int_0^r \bar{\theta} r dr - \frac{E\alpha\bar{\theta}}{1-v} + C_1 \frac{E}{(1+v)(1-2v)} \quad (20)$$

The constant C_1 is found using the boundary condition

$$\sigma_{rr} = 0 \quad \text{at} \quad r = b \quad (21)$$

Which yields $C_1 = \frac{\alpha(1+v)(1-2v)}{(1-v)b^2}\int_0^b \bar{\theta} r dr$ (22)

IV. DETERMINATION OF DISPLACEMENT FUNCTION

Using eq. (16) in eq. (18), we obtain the displacement function as

$$u = \frac{1+v}{1-v}\frac{\alpha}{r}\Bigg[\int_0^r \Bigg(\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n\cos n\phi + B_n \sin n\phi)\Bigg[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +) + \beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm} J_n(\mu_{nm\rho}) \frac{\exp[-(\mu_{nm}^2 + \beta_k^2)\tau]}{(\mu_{nm}^2 + \beta_k^2)\{[n(n+\bar{h})/\mu_{nm} - \mu_{nm}]J_n(\mu_{nm}) - \bar{h}J_{n+1}(\mu_{nm})\}}\Bigg]\Bigg) r dr + (1 - 2v)\frac{r^2}{b^2}\int_0^b \Bigg(\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n\cos n\phi + B_n \sin n\phi)\Bigg[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +) + \beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm} J_n(\mu_{nm\rho}) \frac{\exp[-(\mu_{nm}^2 + \beta_k^2)\tau]}{(\mu_{nm}^2 + \beta_k^2)\{[n(n+\bar{h})/\mu_{nm} - \mu_{nm}]J_n(\mu_{nm}) - \bar{h}J_{n+1}(\mu_{nm})\}}\Bigg]\Bigg) r dr\Bigg]$$

(1) (23)

V. DETERMINATION OF STRESS FUNCTIONS

Using eq. (16) in eq. (20), we obtain the stress function as

$$\sigma_{rr} = \frac{E\alpha}{1-v}\left[\frac{1}{b^2}\int_0^b (\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +)+\beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm\rho})\frac{\exp\left[-(\mu_{nm}^2+\beta_k^2)\tau\right]}{(\mu_{nm}^2+\beta_k^2)\{[\,n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right])rdr - \frac{1}{r^2}\int_0^r (\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +)+\beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm\rho})\frac{\exp\left[-(\mu_{nm}^2+\beta_k^2)\tau\right]}{(\mu_{nm}^2+\beta_k^2)\{[\,n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right])rdr\right]$$

$$\sigma_{rr} = \frac{E\alpha}{1-v}\left[\frac{1}{b^2}\int_0^b (\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +)+\beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm\rho})\frac{\exp\left[-(\mu_{nm}^2+\beta_k^2)\tau\right]}{(\mu_{nm}^2+\beta_k^2)\{[\,n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right])rdr + \frac{1}{r^2}\int_0^r \left(\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +)+\beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm\rho})\frac{\exp\left[-(\mu_{nm}^2+\beta_k^2)\tau\right]}{(\mu_{nm}^2+\beta_k^2)\{[\,n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right]\right)rdr - (\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +)+\beta_k I_{n+1}(\beta_k)} + \sum_{m=1}^{\infty} 2\mu_{nm}J_n(\mu_{nm\rho})\frac{\exp\left[-(\mu_{nm}^2+\beta_k^2)\tau\right]}{(\mu_{nm}^2+\beta_k^2)\{[\,n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right])\right] \tag{24}$$

The stress in axial direction σ_{zz} , is obtained as

$$\sigma_{zz} \frac{E\alpha}{1-v}\left[\frac{2v}{b^2}\int_0^b\left(\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi\right.\right.$$
$$+ B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +) + \beta_k I_{n+1}(\beta_k)}\right.$$
$$\left.\left.+\sum_{m=1}^{\infty} 2\mu_{nm} J_n(\mu_{nm\rho}) \frac{\exp[-(\mu_{nm}^2+\beta_k^2)\tau]}{(\mu_{nm}^2+\beta_k^2)\{[n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right]\right) rdr$$
$$-(\sum_{k=1}^{\infty} C_k \cos\beta_k\xi \sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)\left[\frac{I_n(\beta_k\rho)}{(n+\bar{h})I_n(\beta_k +) + \beta_k I_{n+1}(\beta_k)}\right.$$
$$\left.\left.+\sum_{m=1}^{\infty} 2\mu_{nm} J_n(\mu_{nm\rho}) \frac{\exp\,[-(\mu_{nm}^2+\beta_k^2)\tau]}{(\mu_{nm}^2+\beta_k^2)\{[n(n+\bar{h})/\mu_{nm}-\mu_{nm}]J_n(\mu_{nm})-\bar{h}J_{n+1}(\mu_{nm})\}}\right])\right]$$

VI. CONCLUSION

In this problem, we have investigated the temperature distribution and unknown temperature gradient on outer curved surface of an elliptical cylinder with the aid of the Mathiue function and integral transform technique. The results are obtained in the form of infinite series .The expressions that are obtained can be applied to the design of useful structures or machines in engineering application.

REFERENCES

[1] Choubey, N.K and Shrivastava Geeta, "Inverse Problem of heat conduction for elliptical Cylinder" , Varahmihir Journal of Mathematics Vol.4, No.2, pp. 517-522, 2004.

[2] Mclachaln , N.W, "Theory and Application of Mathieu Function" , Dover Publication, 1964.

[3] Mehta, D.K, Proc. Nat Acad. Sci., 37(2), 1967, 187-190.

[4] Patel S.R, "Inverse Problem of Transient Heat conduction with Radiation", the Mathematics Education Vol.5, No.4, pp.85-90, 1971.

[5] Sneddon I.N, "Fourier Transform", Mc Graw –hill Book Co. Inc, Chap 3, 1951.